than for earth. This follows because of the greater percentage change in lunar gravitational acceleration compared to that of earth over a given altitude increment, and points up the need for special care in applying a constant-g approximation for lunar vertical trajectories.

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Rapid Estimation of the Far-Field in an Axisymmetric Compressible Jet

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It is of interest to be capable of rapidly estimating the far field characteristics of free jets. Hence, the somewhat familiar integral approach, assuming the form of the radial profiles, has been used.[1]

Over-all momentum, energy, and mass balances result in the following equations (assuming that the jet exit static pressure is equal to ambient pressure and that the exit profiles are flat):

$$\pi \int_0^\infty \rho u(u - u_\infty) d(r^2) = \rho_e u_e (u_e - u_\infty) \frac{\pi d^2}{4} \quad (1)$$

$$\pi \int_0^\infty \rho u \left[c(T - T_\infty) + \frac{J}{2}(u^2 - u_\infty^2) \right] d(r^2) = \frac{\pi d^2}{4} \rho_e u_e \left[c_e(T_e - T_\infty) + \frac{J}{2}(u_e^2 - u_\infty^2) + \Delta H_c \right] \quad (2)$$

$$\pi \int_0^\infty \frac{R - R_\infty}{R_e - R_\infty} \rho u d(r^2) = \pi \int_0^\infty \frac{c - c_\infty}{c_e - c_\infty} \rho u d(r^2) = \frac{\pi d^2}{4} \rho_e u_e \quad (3)$$

where c and R are the specific heat at constant pressure and the gas constant, respectively, and ΔH_c, the heat of combustion per pound of exhaust products, is included to account for gross effects due to afterburning (J = conversion factor). The following equations[2, 3] for the variation of the half-radius $r_{1/2}$ (radius at which the velocity defect, $u - u_\infty$, is half of that on the jet centerline),

$$\frac{r_{1/2}}{d} = C_1 \left(\frac{x - a}{d} \right) \qquad \frac{u - u_\infty}{u_\infty} \gg 1 \quad (4a)$$

$$\frac{r_{1/2}}{d} = C_1 \left(\frac{x - a}{d} \right)^{1/3} \qquad \frac{u - u_\infty}{u_\infty} \ll 1 \quad (4b)$$

coupled with radial profiles represented by the Gaussian error law,

$$\frac{u - u_\infty}{u_m - u_\infty} = \exp\left[-\left(\frac{r}{r_{1/2}} \right)^2 \ln 2 \right] \quad (5)$$

$$\frac{T - T_\infty}{T_m - T_\infty} = \frac{R - R_\infty}{R_m - R_\infty} = \frac{c - c_\infty}{c_m - c_\infty} = \exp\left[-\frac{1}{n} \left(\frac{r}{r_{1/2}} \right)^2 \ln 2 \right] \quad (6)$$

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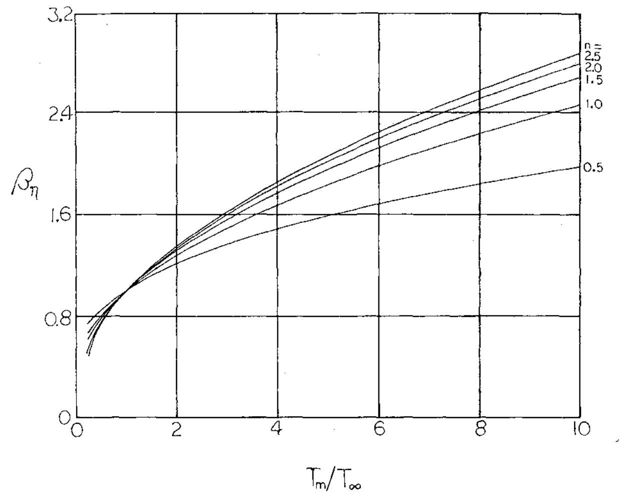

Fig. 1 Nonisothermal correction factor vs temperature ratio with n as a parameter.

yield

$$\frac{u - u_\infty}{u_m - u_\infty} = \exp\left[-\frac{\ln 2}{C_1^2} \left(\frac{r}{x - a} \right)^2 \right] = \exp(-C_2 \xi^2) \qquad \frac{u - u_\infty}{u_\infty} \gg 1 \quad (7a)$$

$$\frac{u - u_\infty}{u_m - u_\infty} = \exp\left[-\frac{\ln 2}{C_1^2} \left(\frac{x - a}{d} \right)^{4/3} \left(\frac{r}{x - a} \right)^2 \right] = \exp\left[-C_2 \left(\frac{x - a}{d} \right)^{4/3} \xi^2 \right] \qquad \frac{u - u_\infty}{u_\infty} \ll 1 \quad (7b)$$

where $C_2 = (\ln 2)/C_1^2$ and $\xi = r/(x - a)$. Similar expressions for the scalar quantities in Eq. (1) result, except for the n, which accounts for the more rapid spread (i.e., $n > 1$) of the scalar quantities as compared to the momentum spread in free jet flow. It is to be noted that n is the reciprocal of the turbulent Prandtl number.

Because of space limitations, only case a for the velocity defect $u - u_\infty$ much larger than the freestream velocity U_∞ will be considered herein. Further results may be found in Ref. 1.

Substituting Eq. (7a), together with its scalar counterparts, into Eqs. (1–3), using the perfect gas law, rearranging, and integrating gives [neglecting the square of $u_\infty/(u_m - u_\infty)$ compared to unity in the energy equation] the following:

Momentum

$$\frac{u_m - u_\infty}{[u_e(u_e - u_\infty)]^{1/2}} \frac{x - a}{d} = \left[\frac{(C_2/2)(R_\infty T_\infty / R_e T_e)}{1 + 2(\beta_n^2/\beta^2_{n/2})[u_\infty/(u_m - u_\infty)]} \right]^{1/2} \beta_n \quad (8)$$

Energy

$$\frac{x - a}{d} = \frac{A}{2} \left[1 + \left(1 - \frac{4B}{A^2} \right)^{1/2} \right] \quad (9)$$

where

$$A = \frac{D}{2n\beta_n} \left[\left(\frac{C_2}{2} \right) \left(\frac{R_\infty T_\infty}{R_e T_e} \right) \left(\frac{u_e}{u_e - u_\infty} \right) \right]^{1/2} \times \frac{c_e}{c_\infty} \frac{T_e - T_\infty}{T_m - T_\infty} \left\{ 1 + \frac{\Delta H_e}{c_e(T_e - T_\infty)} + \frac{J u_e^2}{2c_e(T_e - T_\infty)} \left(1 + 2\frac{u_\infty^2}{u_e^2} - 3\frac{u_\infty}{u_e} \right) \right\} \quad (9a)$$

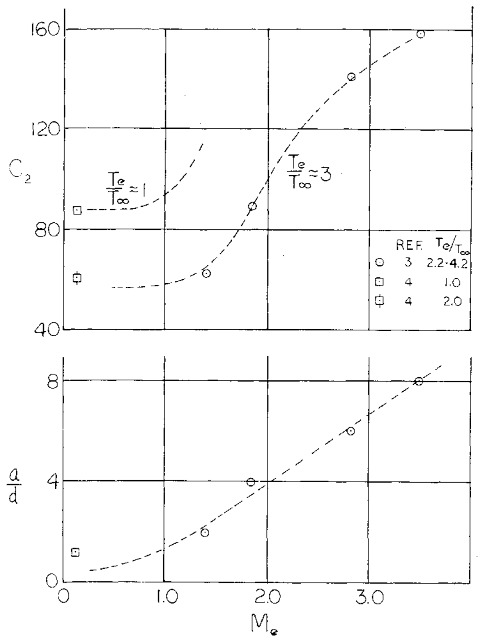

Fig. 2 Similarity constants vs jet exit Mach number.

$$B = \frac{D\beta_n{}^2}{3n\beta^2{}_{3n/2}}\left(\frac{C_2}{2}\right)\left(\frac{R_\infty T_\infty}{R_e T_e}\right)\left(\frac{u_e - u_\infty}{u_e}\right) \times \frac{J u_e{}^2}{2c_e(T_e - T_\infty)}\,\frac{c_e}{c_\infty}\,\frac{T_e - T_\infty}{T_m - T_\infty} \qquad (9b)$$

and

$$\frac{1}{D} = \frac{1}{(n+1)\beta^2{}_{(n+1)/2}} + \frac{(c_m/c_\infty) - 1}{(n+2)\beta^2{}_{(n+2)/2}} + \left(\frac{1}{\beta^2{}_{1/2}} + \frac{(c_m/c_\infty) - 1}{2\beta_1{}^2}\right)\frac{u_\infty}{[u_e(u_e - u_\infty)]^{1/2}} \times \frac{1}{\beta_n}\left(\frac{x-a}{d}\right)\left[\left(\frac{C_2}{2}\right)\left(\frac{R_\infty T_\infty}{R_e T_e}\right)\right]^{-1/2} \qquad (9c)$$

Mass

$$\frac{R_m - R_\infty}{R_e - R_\infty}\,\frac{x-a}{d} = \frac{\frac{1}{2}\left(\frac{C_2}{2}\right)\left(\frac{R_\infty T_\infty}{R_e T_e}\right)}{\left[\left(\frac{C_2}{2}\right)\left(\frac{R_\infty T_\infty}{R_e T_e}\right)\left(\frac{u_e - u_\infty}{u_e}\right)\right]^{1/2}\frac{n}{n+1}\,\frac{\beta_n}{\beta^2{}_{(n+1)/2}} + \frac{x-a}{d}\,\frac{u_\infty}{u_e}\,\frac{n}{\beta^2{}_{1/2}}} \qquad (10)$$

The factor β_n is defined by

$$\frac{1}{\beta_n{}^2} = 2C_2\int_0^\infty \frac{\exp\,(-2C_2\xi^2)\,d(\xi^2)}{\left[1 + \left(\frac{R_m}{R_\infty} - 1\right)\exp\left(-\frac{C_2}{n}\,\xi^2\right)\right]\left[1 + \left(\frac{T_m}{T_\infty} - 1\right)\exp\left(-\frac{C_2}{n}\,\xi^2\right)\right]} = 2n\int_0^1 \frac{y^{(2n-1)}dy}{\left[1 + \left(\frac{R_m}{R_\infty} - 1\right)y\right]\left[1 + \left(\frac{T_m}{T_\infty} - 1\right)y\right]} \qquad (11)$$

which may easily be integrated for integer values of $2n$ to give the result that

$$\frac{1}{\beta_n{}^2} = \frac{n}{n-1}\,\frac{1}{[(R_m/R_\infty) - 1]}\,\frac{1}{[(T_m/T_\infty) - 1]} \times \left[1 - \frac{1}{\beta^2{}_{n-1}} - \left(\frac{R_m}{R_\infty} + \frac{T_m}{T_\infty} - 2\right)\frac{2(n-1)}{2n-1}\,\frac{1}{\beta^2{}_{(2n-1)/2}}\right] \qquad (12)$$

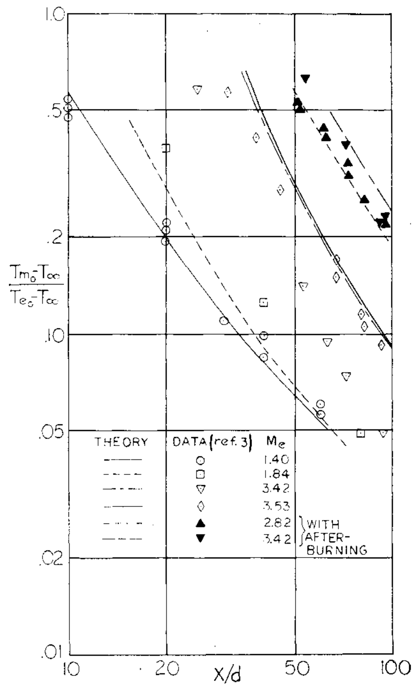

Fig. 3 Comparison of centerline total temperature decay with theory.

and

$$\frac{1}{\beta^2{}_{1/2}} = \frac{1}{(T_m/T_\infty) - (R_m/R_\infty)}\ln\left(\frac{T_m}{T_\infty}\,\frac{R_\infty}{R_m}\right) \qquad (12a)$$

$$\frac{1}{\beta_1{}^2} = \frac{2}{(T_m/T_\infty) - (R_m/R_\infty)} \times \left[\frac{\ln(R_m/R_\infty)}{(R_m/R_\infty) - 1} - \frac{\ln(T_m/T_\infty)}{(T_m/T_\infty) - 1}\right] \qquad (12b)$$

A plot of β_n for the case of $R_m/R_\infty = 1$ is given in Fig. 1. (In Ref. 1, β_n is plotted vs T_m/T_∞ for R_m/R_∞ from 0.5 to 10.) In Fig. 2 is included a rather sparse sampling of data for C_2 and a/d as deduced from $r_{1/2}$ plots in Refs. 3 and 4. It should be noted that Ref. 3 used conical nozzles. It is rather interesting to compare qualitatively C_2 with the jet spread parameter σ used in *near-field* free jet mixing. Both C_2 and σ increase with increasing Mach number, indicating less turbulent mixing, but decrease with increasing exit to ambient temperature ratio, indicating more turbulent mixing.

For a typical rocket situation in which R_e/R_∞ is of the order of unity, the far-field characteristics may be determined as follows. For a given exit Mach number and exit temperature ratio, estimate C_2 and a/d from Fig. 2. Then, pick a value of T_m/T_∞ ($<T_e/T_\infty$), find the β values from Fig. 1, and use Eqs. (9) to find $(x - a)/d$. This will most likely take only one iteration, since the term in D involving $(x - a)/d$ has only a small contribution. Next, Eq. (8) may be used to calculate U_m. The radial profiles may then be found from Eq. (7) and its scalar counterpart. It is recommended to use a value of 1.5 for n. For the case of R_e/R_∞ greatly different from unity, it is necessary to guess a value of R_m/R_∞, along with picking a T_m/T_∞, to get β values. Then, after using Eqs. (9) with the relation $(c_m - c_\infty)/(c_e - c_\infty) = (R_m - R_\infty)/(R_e - R_\infty)$ to get

$(x - a)/d$, Eq. (10) is used to calculate R_m/R_∞, which must be the same as the value picked. This is iterated (once or twice) until the R_m/R_∞ values are the same.

The method given in the foregoing has been compared to the centerline total temperature decay data of Ref. 3 as shown in Fig. 3. For afterburning, ΔH_c was taken as 2200 Btu/lb. These analytical results are seen to be in relatively good agreement with the data.

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A "Membrane" Solution for Axisymmetric Heating of Dome-Shaped Shells of Revolution

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THE following note presents an order of magnitude analysis which leads to a particular solution of the problem of axisymmetric heating of dome-shaped shells of revolution. Such a solution might be termed an approximate analog to Goodier's method in thermoelasticity. The complete solution is constructed by superposing on the particular solution a solution for an edge-loaded shell which is properly adjusted to satisfy the particular boundary conditions of the given problem. This latter solution has been studied extensively, and many approximate solutions are available. The resulting composite solution has the advantage over the more direct methods that the accuracy of the solution in the interior of the shell is quite high.

General Equation

With the notation of Ref. 1 (see Fig. 1), the inclusion of thermal effects modifies the stress-strain law as follows:

$$\sigma_\phi = E\cdot(\epsilon_\phi + \nu\epsilon_\theta)/(1-\nu^2) - \alpha ET/(1-\nu)$$
$$\sigma_\theta = E\cdot(\epsilon_\theta + \nu\epsilon_\varphi)/(1-\nu^2) - \alpha ET/(1-\nu)$$
$$\tau_{\varphi\theta} = G\gamma_{\varphi\theta}$$

If the temperature variations to be studied are limited to

$$T^{(1)} = S(\varphi) \qquad T^{(2)} = (\zeta/h)\cdot\sigma(\varphi) \quad (-h/2 \leqslant \zeta \leqslant h/2)$$

i.e., either middle surface heating or a temperature gradient, and if the functions $S(\varphi)$, $\sigma(\varphi)$ are slowly varying functions of position, one can develop an order of magnitude analysis for each case on the assumption that the deformation variables will also be slowly varying functions of position. Specifically, derivatives with respect to arc length will be considered to be of the order of magnitude of the quality differentiated divided by a length (R_c) that will be taken equal to a measure of the radius of curvature (R_2), i.e.,

$$\frac{d(N,M)}{ds} = \frac{(N,M)}{R_c}\,0(1) \qquad \frac{R_2}{R_c} = 0(1)$$

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Upon introducing the following nondimensional variables

$$S(\varphi) = S_c\cdot\tilde{S} \qquad \sigma(\varphi) = \sigma_c\tilde{\sigma}$$
$$\tilde{\epsilon}_i^{(1)} = \frac{\epsilon_{i0}}{\alpha S_c} \qquad \tilde{\epsilon}_i^{(2)} = \frac{12R_c\epsilon_{i0}}{\sigma_c\alpha h}$$
$$\tilde{V}^{(1)} = \frac{V}{\alpha S_c} \qquad \tilde{V}^{(2)} = \frac{12R_cV}{\sigma_c\alpha h}$$
$$\tilde{N}_{ii}^{(1)} = \frac{N_{ii}}{Eh\alpha S_c} \qquad \tilde{N}_{ii}^{(2)} = \frac{12(1-\nu)R_cN_{ii}}{\sigma_c\alpha Eh^2}$$
$$\tilde{M}_{ii}^{(1)} = \frac{M_{ii}}{M_T} \qquad \tilde{M}_{ii}^{(2)} = \frac{12(1-\nu)M_{ii}}{\sigma_c\alpha Eh^2}$$
$$\tilde{Q}_\varphi^{(1)} = \frac{Q_\varphi R_c}{M_T} \qquad \tilde{Q}_\varphi^{(2)} = \frac{12(1-\nu)R_cQ_\varphi}{\sigma_c\alpha Eh^2}$$
$$M_T = \frac{\alpha Eh^3S_c}{12(1-\nu)R_c} \qquad \tilde{R}_2 = \frac{R_2}{R_c}$$
$$\tilde{s} = \frac{s}{R_c} \qquad \epsilon = \frac{h^2}{12R_c^2}$$
$$\lambda = \frac{R_2}{R_1}$$

in the augmented relations of Ref. 1, we obtain a set of equations for each type of temperature variation. They are: the equations of equilibrium

$$\tilde{Q}_\varphi^{(1,2)} = (d\tilde{M}_{\varphi\varphi}^{(1,2)}/d\tilde{s}) + (\tilde{M}_{\varphi\varphi}^{(1,2)} - \tilde{M}_{\theta\theta}^{(1,2)})\cdot\cot\varphi/\tilde{R}_2$$
$$\tilde{N}_{\phi\phi}^{(1,2)} = [\epsilon/(1-\nu),\ 1]\cdot\tilde{Q}_\varphi^{(1,2)}\cdot\cot\varphi$$
$$\tilde{N}_{\theta\theta}^{(1,2)} = [\epsilon/(1-\nu),\ 1]\cdot(d/d\tilde{s})\ (R_2\tilde{Q}_\varphi^{(1,2)})$$

the equation of compatibility

$$\tilde{V}^{(1,2)} = -\tilde{R}_2\cdot\frac{d\tilde{\epsilon}_\theta^{(1,2)}}{d\tilde{s}} + (\tilde{\epsilon}_\varphi^{(1,2)} - \tilde{\epsilon}_\theta^{(1,2)})\cdot\cot\varphi$$

and, the stress-strain, moment-curvature relations

$$[(1-\nu^2)\tilde{N}_{\phi\phi}^{(1)},\ (1+\nu)\tilde{N}_{\varphi\varphi}^{(2)}] = \tilde{\epsilon}_\varphi^{(1,2)}\cdot(1 - \epsilon\lambda/\tilde{R}_2^2) + \nu\tilde{\epsilon}_\theta^{(1,2)}(1 - \epsilon/\tilde{R}_2^2) - (1+\nu)\cdot(\tilde{S},\ \tilde{\sigma}/\tilde{R}_2) + (\epsilon/\tilde{R}_2)\cdot[(d\tilde{V}^{(1,2)}/d\tilde{s}) + \nu\cdot\tilde{V}^{(1,2)}\cot\varphi/\tilde{R}^2]$$

$$[(1-\nu^2)\cdot\tilde{N}_{\theta\theta}^{(1)},\ (1+\nu)\tilde{N}_{\theta\theta}^{(2)}] = \tilde{\epsilon}_\theta^{(1,2)}\cdot(1 - \lambda\epsilon/\tilde{R}_2^2) + \nu\tilde{\epsilon}_\varphi^{(1,2)}\cdot(1 - \lambda^2\epsilon/\tilde{R}_2^2) - (1+\nu)(\tilde{S},\ \lambda\tilde{\sigma}/\tilde{R}_2) + (\lambda\epsilon/\tilde{R}_2^2)\cdot[\tilde{V}^{(1,2)}\cot\varphi + \nu\tilde{R}_2\cdot(d\tilde{V}^{(1,2)}/d\tilde{s})]$$

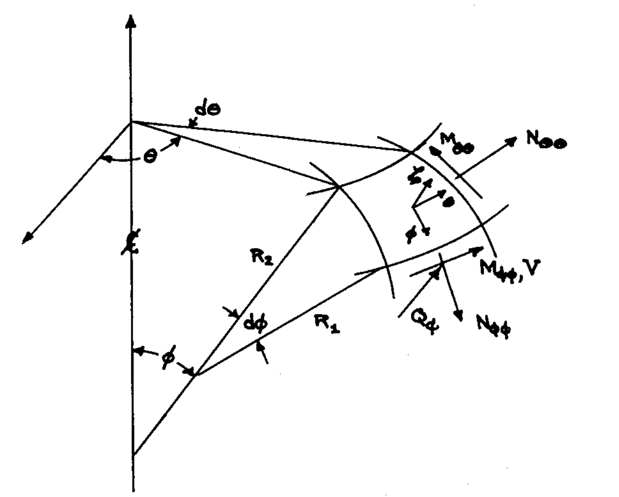

Fig. 1 Notation.